Chapter 5.

Geometric proofs.

Proof

An attempt to prove a statement to be true involves establishing and presenting sufficient evidence to convince others of the fact that the statement is true. In a courtroom situation the prosecuting team attempts to present sufficient evidence to convince the members of the jury that the defendant is guilty. The defence team, on the other hand, attempts to cause the jury members to doubt the claim that the defendant is guilty.

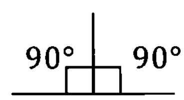
Similarly, in mathematics if we wish to prove a statement true, for example that the angles of a triangle add up to 180°, we must present evidence to convince others of the truth of the statement. The Preliminary work, and some questions in Miscellaneous Exercises One and Two involved the use of similar triangles and congruent triangles to prove various things to be true.

Definition, axioms and theorems.

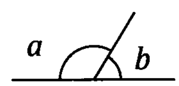
Some statements do not need to be proved because they are true *by definition.*

For example, if we define 1 centimetre (1 cm) to be one 100^{th} part of a metre it follows from this definition that 100 cm = 1 metre.

If we define 1 degree (1°) to be one 90^{th} part of a right angle it follows from this definition that 90° = 1 right angle.



From basic definitions other true statements can follow. For example, if we define the angle of a straight line to equal two right angles it follows that with a and b as shown in the diagram on the right



$$a + b = \text{Two right angles}$$

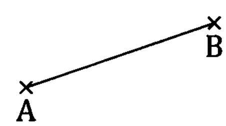
and so

$$a + b = 2 \times 90°$$
$$= 180°$$

There are some statements that are simply accepted as being true without the need for proof. Such statements are called **axioms.**

For example we accept the statement

There is only one straight line that can be drawn to join two specific points in space



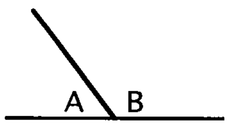
as being true without proof.

The statement is an axiom. It is simply accepted as fact.

Theorems are statements that can be proved to be true using accepted definitions, axioms and other (proven) theorems. The truth of the theorem is arrived at by reasoning from other accepted truths. It is deduced. In this **proof by deduction** the validity of the proof depends upon the correctness of the axioms and theorems used to deduce it.

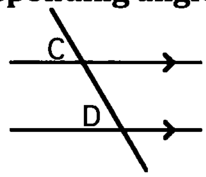
For example, suppose we accept the following two statements as fact:

Angles that together form a straight line have a sum of 180°.



A + B = 180°

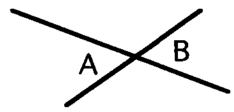
When a transversal cuts parallel lines, corresponding angles are equal.



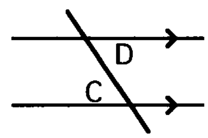
C = D

It is then possible to prove the following statements true, for example:

- ☛ When two straight lines intersect the vertically opposite angles are equal.
  I.e., in the diagram on the right, A = B.



- ☛ When a transversal cuts parallel lines, alternate angles are equal.
  I.e., in the diagram on the right, C = D.

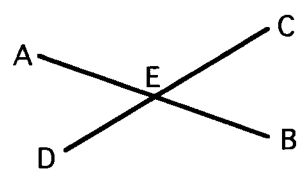


The proofs of these two statements, given the bold statements at the top of the page, follow as examples 1 and 2:

Example 1

Proof of: When two straight lines intersect the vertically opposite angles are equal.

Given: Two straight lines AB and CD intersecting at E.



To prove: Vertically opposite angles are equal.
I.e., in the diagram on the right,

$$\angle AED = \angle BEC$$

and

$$\angle AEC = \angle DEB.$$

Proof:

	∠AED + ∠AEC	=	180°	(Angle sum of straight line DC.)
∴	∠AED	=	180° – ∠AEC	
	∠BEC + ∠AEC	=	180°	(Angle sum of a straight line AB.)
∴	∠BEC	=	180° – ∠AEC	
Hence	∠AED	=	∠BEC as required.	(Each equal to 180° – ∠AEC.)
Also	∠AEC	=	180° – ∠AED	(Angle sum of straight line DC.)
	∠DEB	=	180° – ∠AED	(Angle sum of straight line AB.)
Hence	∠AEC	=	∠DEB as required.	(Each equal to 180° – ∠AED.)

Notice that as was the case with the proofs given in the preliminary work, the previous proof features

a clear statement of what is "given",

a statement of what it is that we are attempting "to prove",

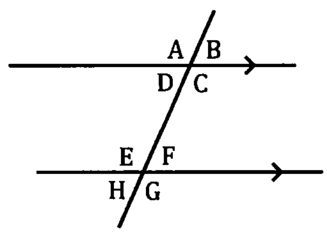
a clear diagram,

and that known truths are referred to in order to justify a statement.

Note also that as was mentioned in the Preliminary work section, these justifications do not need to be essays but are instead brief statements that clearly indicate which truth justifies the statement.

Example 2

Proof of: When a transversal cuts parallel lines, alternate angles are equal.



Given: A transversal cutting a pair of parallel lines with angles A, B, C, D, E, F, G and H as shown in the diagram on the right.

To prove: Alternate angles are equal.
I.e., in the diagram on the right, C = E
and D = F.

Proof:

	C = G	(Corresponding angles are equal.)
But	E = G	(Vertically opposite angles are equal, just proved.)
∴	C = E, as required.	
Similarly	D = H	(Corresponding angles are equal.)
But	F = H	(Vertically opposite angles are equal, just proved.)
∴	D = F, as required.	

☞ Some proofs may end with the abbreviation QED rather than the "as required" statement used above. QED stands for

quad erat demonstrandum

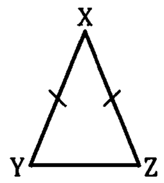
which is a Latin phrase meaning "which was to be demonstrated".

Circle properties.

This chapter concentrates on proving and using various circle properties. In the examples and exercises that follow the properties of straight lines and triangles mentioned, or proven, in earlier pages, may be stated as fact, without proof.

In particular see the list on the next page.

- ☛ Angles that together form a straight line have a sum of 180°.
(And, conversely, if angles have a sum of 180° then they together form a straight line.)
- ☛ When a transversal cuts parallel lines, corresponding angles are equal.
(And conversely, if corresponding angles are equal then we have parallel lines.)
- ☛ When two straight lines intersect the vertically opposite angles are equal.
(And conversely, if vertically opposite angles are equal then the intersecting lines are straight lines.)
- ☛ When a transversal cuts parallel lines, alternate angles are equal.
(And, conversely, if alternate angles are equal then we have parallel lines.)
- ☛ When a transversal cuts parallel lines, co-interior angles are supplementary.
(And conversely, if co-interior angles are supplementary then we have parallel lines.)
- ☛ The angles of a triangle add up to 180°.
(And conversely, if the angles of a polygon sum to 180° then the polygon is a triangle.)
- ☛ The angles of a quadrialteral add up to 360°.
(And conversely, if the angles of a polygon sum to 360° then the polygon is a quadrilateral.)
- ☛ If a triangle has two sides of the same length then the angles opposite these sides are of equal size.
I.e. in the isosceles triangle shown on the right, with XY = XZ then the base angles, ∠XYZ and ∠XZY, are equal.
(And conversely, if a triangle has two angles of the same size then the sides opposite these angles are of equal length.)



Also ☛ any fact proved in a previous question can be used as fact to prove any later question.

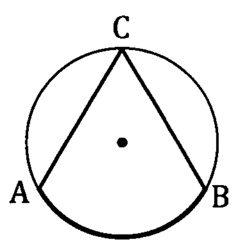
and ☛ don't forget that the concepts of • similar triangles
and • congruent triangles
can be useful for some proofs.

Angles in circles.
The next two examples, and the exercise that follows, considers (and uses) some geometrical results concerning angles in circles.

First though, note the use of the word *subtends*:

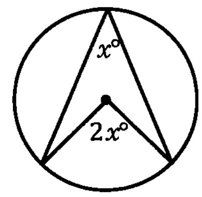
We say that the arc AB shown in the diagram on the right **subtends** the angle ACB at C.

I.e. straight lines drawn from the extremities of the arc, to the point C, form the angle ACB.

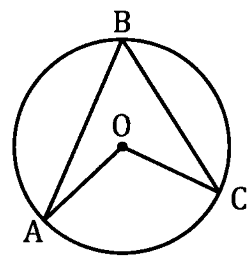


Examples 3 and 4 involve the following result:

> The angle an arc subtends at the centre of a circle is twice the angle the same arc subtends at the circumference.
>
> This is referred to as:
>
> **Angle at the centre is twice the angle at the circumference.**

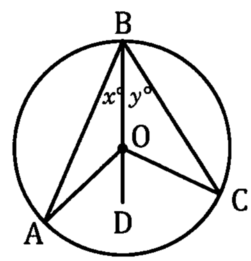


Example 3 (Proof of the above statement.)



Given: Points A, B and C lying on the circumference of a circle centre O, as shown in the diagram.

To prove: In the given diagram $\angle AOC = 2 \times \angle ABC$.



Construction: Draw a line from B to pass through O to some point D. Let $\angle ABO = x°$ and $\angle CBO = y°$.

Proof:

$OA = OB$ (Radii.)

$\therefore$ ΔOAB is isosceles.

Thus $\angle OAB = \angle OBA = x°$ (Base angles of isosceles Δ.)

$\therefore$ $\angle AOB = 180° - 2x°$ (Angle sum of a triangle.)

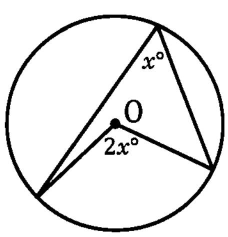
and so $\angle AOD = 2x°$ (Angle of straight line BD.)

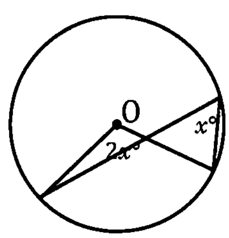
Similarly reasoning for ΔOCB gives

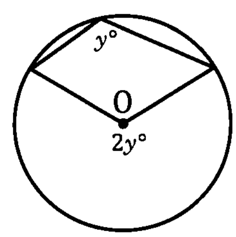
$\angle COD = 2y°$

Now
$$\begin{aligned}\angle AOC &= \angle AOD + \angle COD \\ &= 2x° + 2y° \\ &= 2(x° + y°) \\ &= 2 \times \angle ABC, \text{ as required.}\end{aligned}$$

Note: Situations involving this rule may not always look quite like the diagram above. See the diagrams below for example.







Example 4

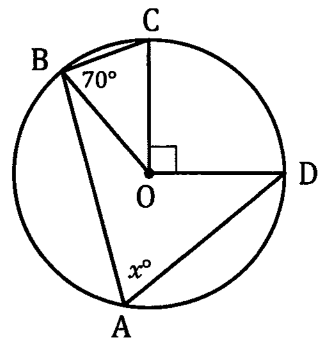
In the diagram on the right point O is the centre of the circle and points A, B, C and D lie on the circle.

$\angle CBO = 70°$,

$\angle COD = 90°$

and $\angle BAD = x°$.

Prove that $x = 65$.



To prove: That for the given diagram, $x = 65$.

Proof: OB = OC (Radii)

$\therefore$ ΔOBC is isosceles.

Thus $\angle OCB = 70°$ (Base angles of isos Δ.)

and $\angle BOC = 40°$ (Angle sum of a triangle.)

Now $\angle BOD = \angle BOC + \angle COD$

$= 40° + 90°$

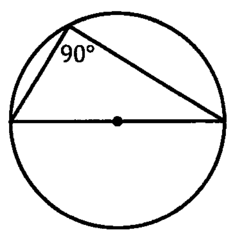
$= 130°$

Hence $\angle BAD = 65°$ (Angle at centre is twice angle at circumference.)

Thus $x = 65$, as required.

Exercise 5A

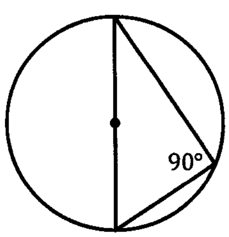
1. Prove the following:

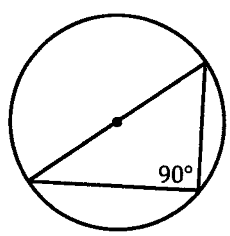


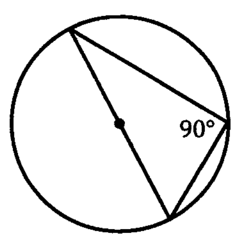
The angle a diameter subtends at the circumference is a right angle.

Or, as we tend to remember it:

Angles in a semicircle are right angles.







2. Prove the following result:
Angles that the minor arc AB subtends at points on the major arc AB are equal (see diagram I below) and, similarly, angles that the major arc AB subtends at points on the minor arc AB are equal (see diagram II below).

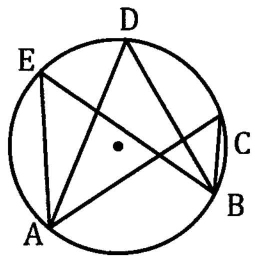


Diagram I
∠AEB = ∠ADB = ∠ACB

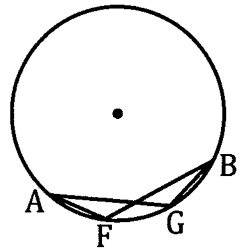


Diagram II
∠AFB = ∠AGB

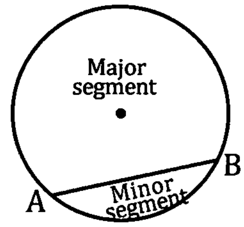


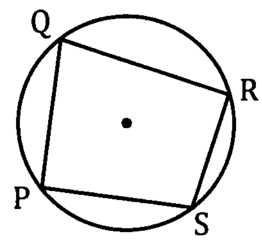
Diagram III

With the Chord AB splitting the circle into a minor segment and a major segment (see diagram III above) we tend to refer to this fact as

Angles in the same segment are equal.

3. If the four vertices of a quadrilateral lie on the circumference of a circle we call the quadrilateral a **cyclic quadrilateral.**
Prove that:

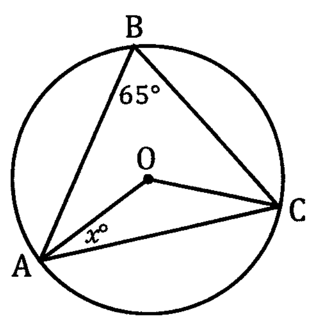
The opposite angles of a cyclic quadrilateral add up to 180°.



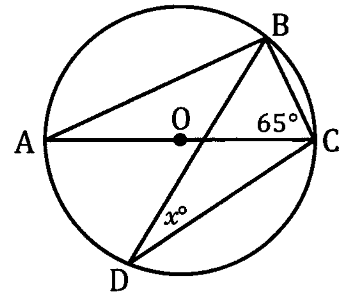
You will find the facts proved in example 3 and questions 1 to 3 of this exercise useful for questions 4 to 9. The facts are summarised below:

The angle at the centre of a circle is twice the angle at the circumference.
Angles in a semicircle are right angles.
Angles in the same segment are equal.
The opposite angles of a cyclic quadrilateral are supplementary.

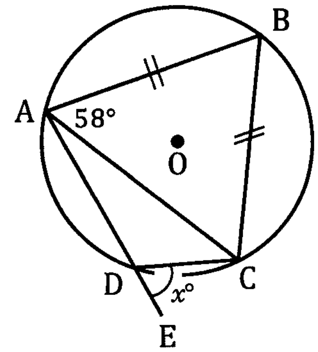
4. In the diagram on the right points A, B and C lie on the circle centre O.
Given that ∠ABC = 65°
and ∠OAC = $x°$
prove that x = 25.



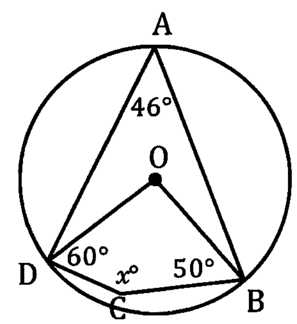
5. In the diagram on the right points A, B, C and D lie on the circle centre O and AC is a diameter of the circle.
Given that $\angle ACB = 65°$
and $\angle BDC = x°$
prove that $x = 25$.



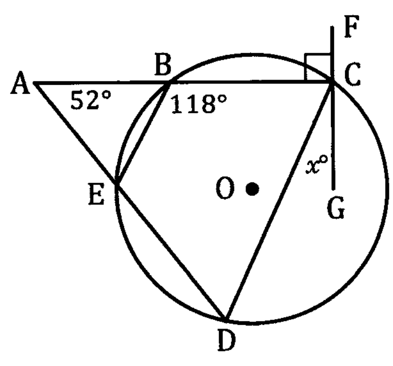
6. In the diagram on the right points A, B C and D lie on the circle centre O.
E is a point outside the circle and ADE is a straight line.
Given that BA = BC,
$\angle BAC = 58°$
and $\angle CDE = x°$
prove that $x = 64$.



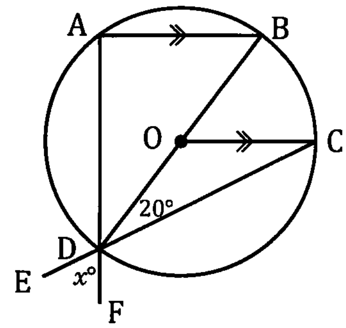
7. In the diagram on the right points A, B and D lie on the circle centre O.
Point C lies inside the circle as shown in the diagram.
Given that $\angle DAB = 46°$,
$\angle ODC = 60°$,
$\angle OBC = 50°$
and $\angle DCB = x°$
prove that $x = 158$.



8. In the diagram on the right points B, C, D and E lie on the circle centre O.
FCG is a straight line and $\angle FCB = 90°$.
CB produced (continued) meets DE produced at point A.
Given that $\angle BAE = 52°$,
$\angle CBE = 118°$
and $\angle DCG = x°$
prove that $x = 24$.



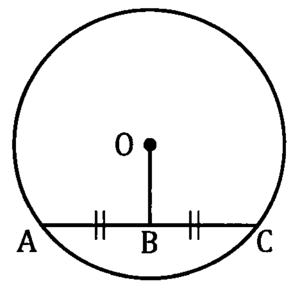
9. In the diagram on the right points A, B, C and D lie on the circle centre O, with BD as a diameter.
F is a point on AD produced (continued).
E is a point on CD produced.
AB is parallel to OC.
Given that ∠ODC = 20°,
and ∠EDF = x°
prove that x = 70.



10. Prove that:

A line drawn from the centre of a circle to the mid-point of any chord of the circle meets that chord at right angles.

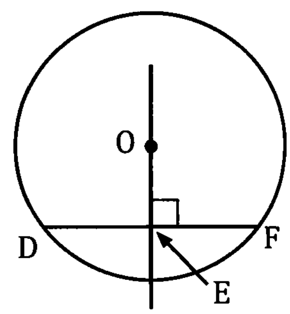
I.e., for the diagram shown on the right prove that
∠OBA = ∠OBC = 90°.



11. Prove that:

If a line is drawn perpendicular to any chord of a circle, and passing through the centre of the circle, then the line will bisect the chord.

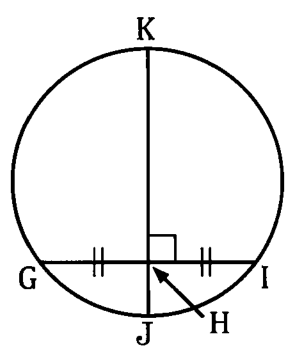
I.e., for the diagram shown on the right prove that
DE = EF.



12. Prove that:

The perpendicular bisector of any chord of a circle passes through the centre of that circle.

I.e., for the diagram shown on the right prove that the line JK consists of points equidistant from G and I and one of these points must be the centre of the circle.



13. Prove that:
Chords of equal length subtend equal angles at the centre.

14. Prove that:
Chords that subtend equal angles at the centre of a circle are equal in length.

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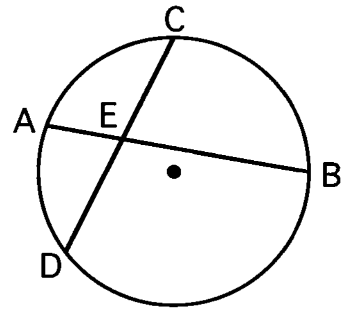
15. Prove that:

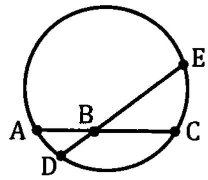
If any two chords of a circle, that we will call AB and CD, intersect at some point, that we will call E, then



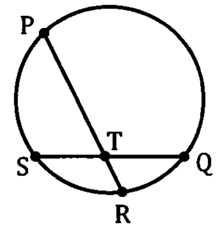
$$AE \times EB = DE \times EC.$$

Hence determine x and y in the following:

AB = 4 cm
AC = 10 cm
BE = 8 cm
DB = x cm

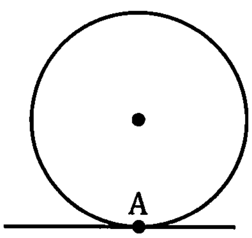


ST = 5 cm
TQ = 6 cm
PR = y cm
PT = 10 cm

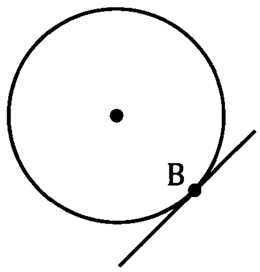


Tangents and secants.

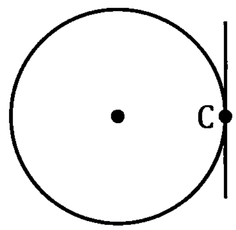
In each of the diagrams below the straight line is a *tangent* to the circle. (From the Latin *tangere*, to touch.) Each tangent just *touches* the circle.



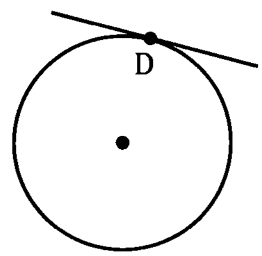
Tangent touches the circle at A.



Tangent touches the circle at B.

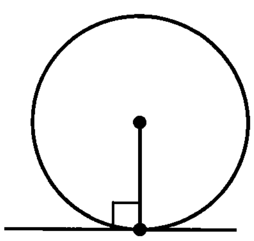


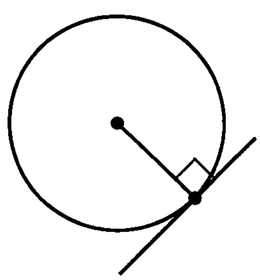
Tangent touches the circle at C.

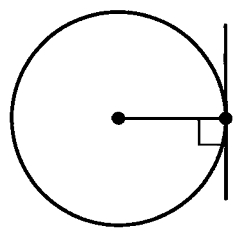


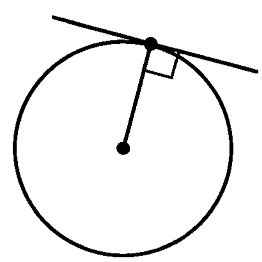
Tangent touches the circle at D.

The angle between a tangent and the radius drawn at the point of contact is a right angle.



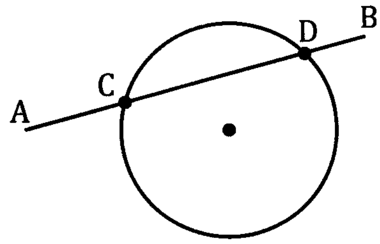






If a straight line cuts a circle at two distinct points it is a **secant** to the circle. (From the Latin *secare*, to cut.)

Thus in the diagram on the right the line AB is a secant of the circle shown, cutting it at points C and D.



The "angle between tangent and radius" statement on the previous page can be used in any questions that follow.

You may also quote, without proof, any of the "circle facts" encountered earlier in this chapter, for example

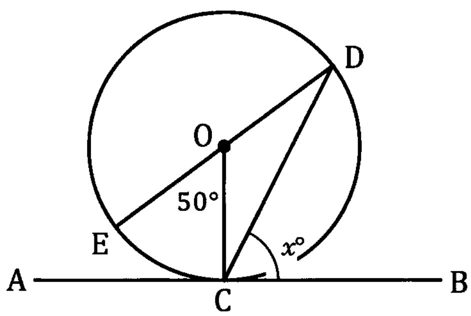
angle at centre = 2 × angle at circumference

(though you should be able to prove this statement if required).

Example 5

In the diagram on the right AB is a tangent to the circle centre O with C the point of contact. ED is a diameter of the circle.

Given that	∠EOC	=	50°,
and	∠DCB	=	x°
prove that	x	=	65.



Given: Diagram as shown.

To prove: $x = 65$

Proof: ∠ODC = 25° (Angle at centre = 2 × angle at circumference.)

ΔOCD is isosceles. (OD and OC are radii.)

∴ ∠OCD = 25° (∠OCD = ∠ODC, base angles of isosceles Δ.)

But ∠OCD + ∠DCB = 90° (Angle between tangent and radius.)

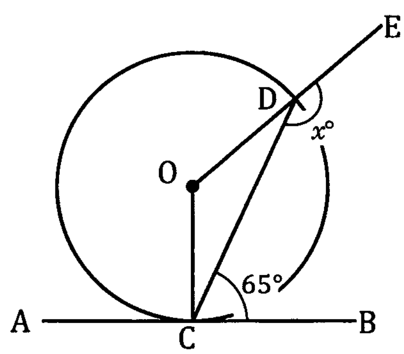
∴ $25 + x = 90$

and so $x = 65$, as required.

Exercise 5B

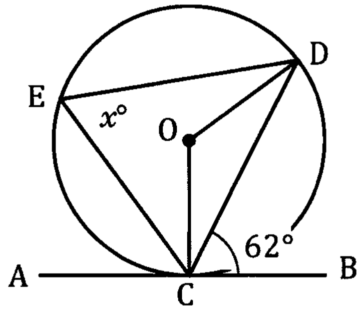
1. In the diagram on the right, AB is a tangent to the circle centre O with C the point of contact. Point D lies on the circle. Point E lies outside the circle with ODE a straight line.

Given that	∠DCB	=	65°,
and	∠EDC	=	x°
prove that	x	=	155.

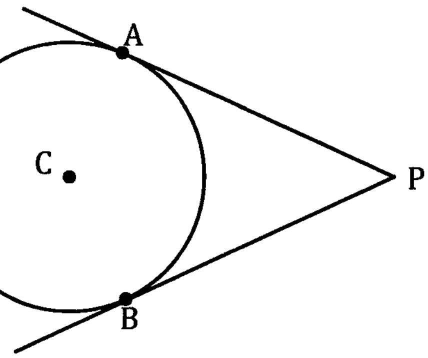


2. In the diagram on the right AB is a tangent to the circle centre O, point C being the point of contact. Points D and E lie on the circle as shown.

   Given that $\angle DCB = 62°$,
   and $\angle CED = x°$
   prove that $x = 62$.

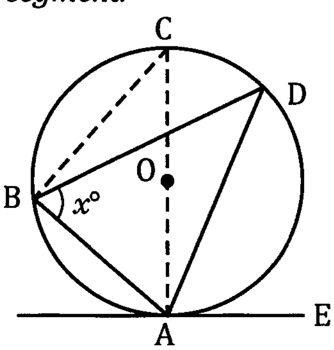


3. The diagram on the left shows point P lying outside a circle centre C with the two tangents from P to the circle shown.
   Prove that:
   (a) the two tangents are of equal length (i.e. that PA = PB),
   (b) PC bisects $\angle APB$ (i.e. that $\angle CPA = \angle CPB$).

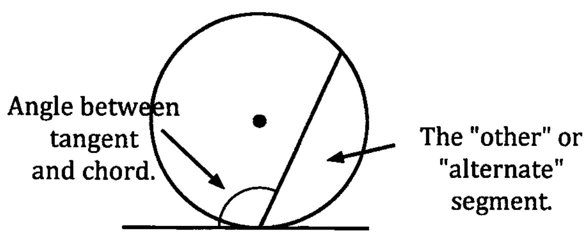


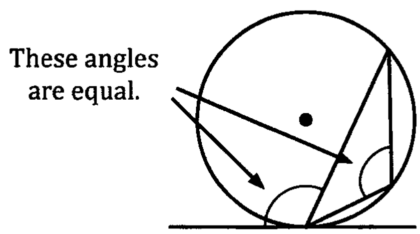
4. (The general case of situation shown in question 2.)
   Prove that: *The angle between a tangent and a chord drawn at the point of contact is equal to the angle in the alternate segment.*

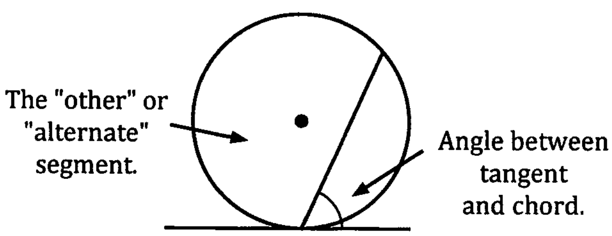
   I.e. for the diagram shown on the right, prove that $\angle DAE = x°$.

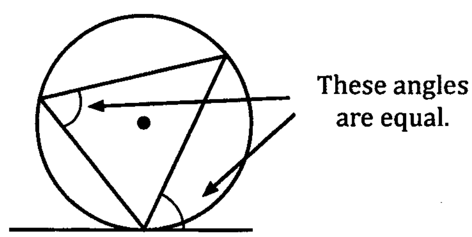


   Note: If you are not sure what is meant by the phrase *the alternate segment* see below:





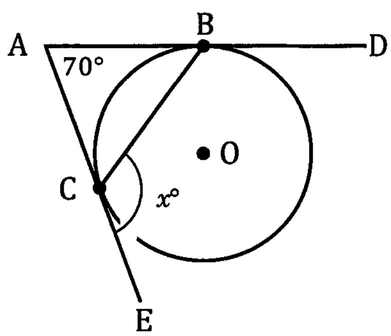




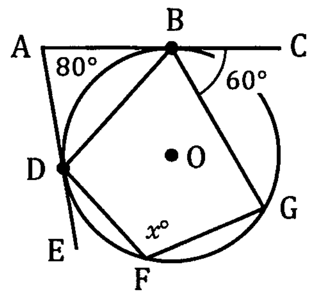
In the remaining questions of this exercise you may use any of the following facts, without proof, in order to justify statements you make (in addition to the standard facts about parallel lines, straight lines, triangles etc.):

The angle at the centre is twice the angle at the circumference.
Angles in a semicircle are right angles.
Angles in the same segment are equal.
The opposite angles of a cyclic quadrilateral are supplementary.
The angle between a tangent and a radius is a right angle.
The angle between tangent and chord equals angle in alternate segment.
The two tangents drawn from a point to a circle are of equal length.

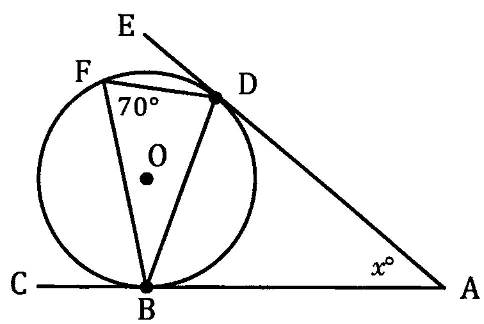
5. In the diagram on the right AD and AE are tangents to a circle centre O, with points of contact B and C respectively.
Given that $\angle DAE = 70°$,
and $\angle BCE = x°$,
prove that $x = 125$.



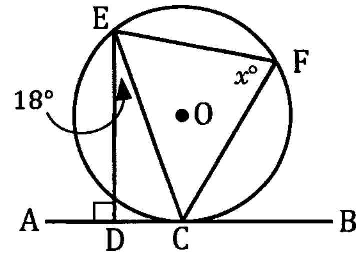
6. In the diagram on the right AC and AE are tangents to a circle centre O, with points of contact B and D respectively. Points F and G lie on the circle,
Given that $\angle DAB = 80°$,
$\angle GBC = 60°$,
and $\angle DFG = x°$,
prove that $x = 110$.



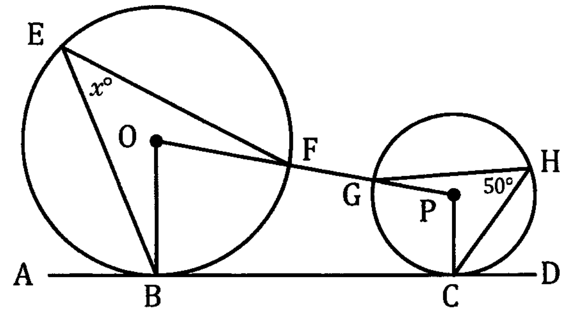
7. In the diagram shown on the right ABC and ADE are tangents to the circle centre O and points, B, D and F lie on the circle.
Given that $\angle DFB = 70°$
and $\angle DAB = x°$
prove that $x = 40$.



8. In the diagram on the right points C, E and F lie on the circle centre O.
ACB is a tangent to the circle and the perpendicular from E to AB meets AB at D.
If ∠DEC = 18° and ∠EFC = x° prove that x = 72.

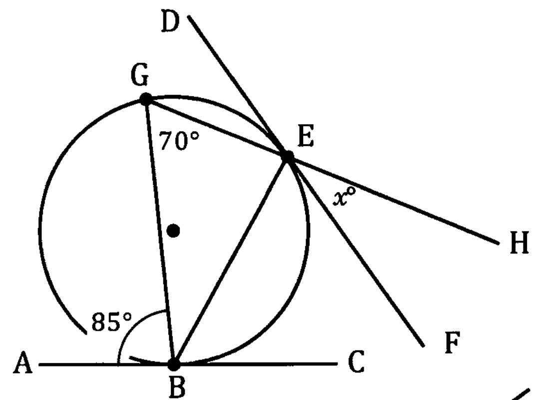


9. The diagram shows two circles.
The larger circle has centre O and the smaller has centre P.
Points B, E and F lie on the larger circle and points C, G and H lie on the smaller circle.
ABCD is a tangent to both circles.
∠GHC = 50° and ∠BEF = x°.
Prove that x = 40.

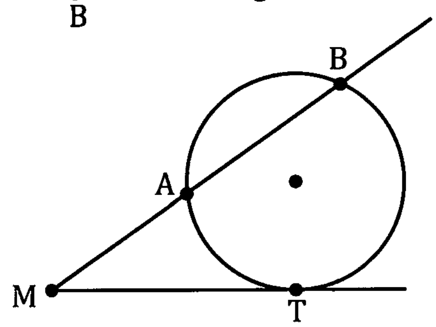


10. The straight lines ABC and DEF are tangents to a circle with points B and E respectively the points of contact with the circle, as shown in the diagram.
A point H lies outside the circle and HE produced (continued) meets the circle again at G.

Given that	∠BGE	=	70°,
	∠GBA	=	85°
and	∠FEH	=	x°
prove that	x	=	25.



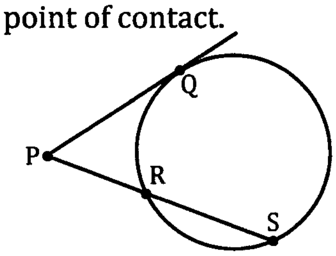
11. The diagram on the right shows a secant cutting a circle at points A and B, and a tangent, touching the circle at T, both drawn from some external point M.
Prove that $MT^2 = MA \times MB$.



Hence determine x and y in the following:

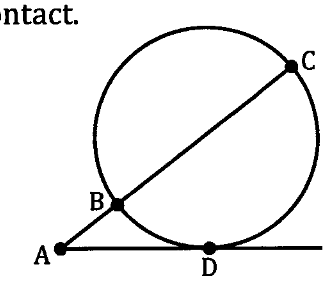
PQ is a tangent to the circle with Q as the point of contact.

PQ = x cm
PR = 8 cm
RS = 10 cm



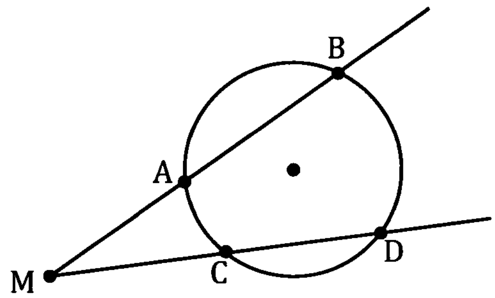
AD is a tangent to the circle with D as the point of contact.

AB = y cm
BC = 30 cm
AD = 20 cm



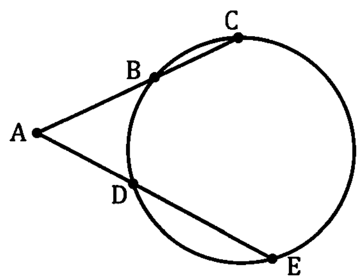
12. The diagram on the right shows two secants to a circle, each drawn from the point M, with one cutting the circle at points A and B and the other cutting the circle at points C and D. Prove that:

$$MA \times MB = MC \times MD.$$

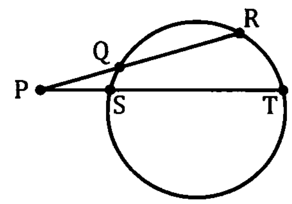


Hence determine x and y in the following:

AB = 40 cm
BC = 60 cm
AD = 50 cm
DE = x cm



PQ = 6 cm
PR = 15 cm
ST = 13 cm
PS = y cm



Miscellaneous Exercise Five.

This miscellaneous exercise may include questions involving the work of this chapter, the work of any previous chapters, and the ideas mentioned in the preliminary work section at the beginning of the book.

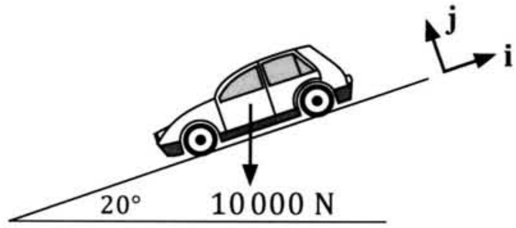
1. A band records 13 new songs and wishes to select 8 of the 13 for release on a forthcoming CD. How many different groups of eight songs can be chosen from the 13?
   Once the selection of the eight songs has been made in how many different orders can these eight occur on the CD?

2. Set A = {a, b, c, d, e, f, g, h, i}. How many different subsets of A are there that consist of six different letters, two of which are vowels and the other four are consonants?

3. How many combinations of six University courses can be chosen from ten courses if two of the ten are compulsory?
   Suppose now that the two compulsory courses are put in a list A, along with three other courses, and the remaining five courses form list B. How many combinations of six courses are there now if the two compulsory courses are still compulsory and three of the six courses must come from List A and three from list B?

4. In still air an aircraft can maintain a speed of 350 km/h. In what direction should the aircraft be pointing if it is to travel on a bearing 170° and a wind of 50 km/h is blowing from 020°.

5. The diagram on the right shows a vehicle parked on a slope, angle of inclination 20°. The weight of the vehicle acts as a downward force of 10 000 N .



(a) With the unit vectors **i** and **j** as shown in the diagram express this 10 000 N force in the form (a**i** + b**j**) N with a and b given to the nearest hundred.

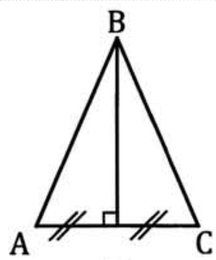
(b) What is the magnitude of the resistance force the brakes must be exerting to prevent the vehicle moving down the slope? (Give your answer to the nearest 100 N.)

6. A teacher asked the class to prove the following:

If in triangle ABC the perpendicular bisector of AC passes through B then ABC is an isosceles triangle.

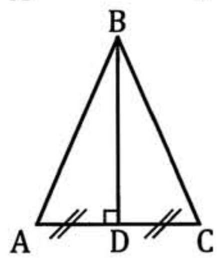
One student presented the following proof:

Given: ΔABC with the perpendicular bisector of AC passing through B.



To prove: That ΔABC is an isosceles triangle.

Proof: Let the mid point of AC be D (see diagram).



In triangles ABD and BDC

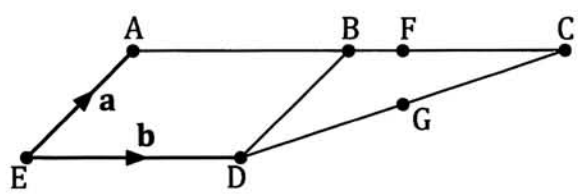
AD	=	DC	(D is mid point of AC.)
BDA	=	BDC	(Each angle equals 90°.)
BD	=	BD	(Common side.)

Thus ΔABD is congruent to ΔBDC (RHS).

Therefore ΔABC is an isosceles triangle.

If the answer were to be marked out of ten how many marks would you award the above response and why?

7. The diagram shows the parallelogram ABDE, with $\overrightarrow{EA} = \mathbf{a}$, and $\overrightarrow{ED} = \mathbf{b}$. AB is produced to C where $\overrightarrow{BC} = \overrightarrow{AB}$. Point F lies on BC and is such that $\overrightarrow{BF} = \frac{1}{3}\overrightarrow{BC}$. G is the mid point of DC.



Find the following vectors in terms of **a** and/or **b**.

(a) $\overrightarrow{AC}$ (b) $\overrightarrow{AF}$ (c) $\overrightarrow{DC}$ (d) $\overrightarrow{EC}$ (e) $\overrightarrow{EG}$ (f) $\overrightarrow{GF}$

If GF and EC intersect at H with $\overrightarrow{GH} = h\,\overrightarrow{GF}$ and $\overrightarrow{HC} = k\,\overrightarrow{EC}$ determine h and k.